Use the imaginary unit $i$ to write complex numbers, and add, subtract, and multiply complex numbers.

- Find complex solutions of quadratic equations.
- Write the trigonometric forms of complex numbers.
- Find powers and $n$th roots of complex numbers.

**Operations with Complex Numbers**

Some equations have no real solutions. For instance, the quadratic equation

$$x^2 + 1 = 0$$

has no real solution because there is no real number $x$ that can be squared to produce $-1$. To overcome this deficiency, mathematicians created an expanded system of numbers using the **imaginary unit** $i$, defined as

$$i = \sqrt{-1}$$

where $i^2 = -1$. By adding real numbers to real multiples of this imaginary unit, you obtain the set of **complex numbers**. Each complex number can be written in the **standard form** $a + bi$. The real number $a$ is called the **real part** of the **complex number** $a + bi$, and the number $bi$ (where $b$ is a real number) is called the **imaginary part** of the complex number.

**Definition of a Complex Number**

For real numbers $a$ and $b$, the number

$$a + bi$$

is a **complex number**. If $b \neq 0$, then $a + bi$ is called an **imaginary number**. A number of the form $bi$, where $b \neq 0$, is called a **pure imaginary number**.

To add (or subtract) two complex numbers, you add (or subtract) the real and imaginary parts of the numbers separately.

**Addition and Subtraction of Complex Numbers**

If $a + bi$ and $c + di$ are two complex numbers written in standard form, then their sum and difference are defined as follows.

**Sum:**

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$

**Difference:**

$$(a + bi) - (c + di) = (a - c) + (b - d)i$$
The additive identity in the complex number system is zero (the same as in the real number system). Furthermore, the additive inverse of the complex number $a + bi$ is $-(a + bi) = -a - bi$. Additive inverse

So, you have

$$(a + bi) + (-a - bi) = 0 + 0i = 0.$$

**EXAMPLE 1** Adding and Subtracting Complex Numbers

a. $(3 - i) + (2 + 3i) = 3 - i + 2 + 3i$

   $$= 3 + 2 - i + 3i$$
   $$= (3 + 2) + (-1 + 3)i$$
   $$= 5 + 2i$$

b. $2i + (-4 - 2i) = 2i - 4 - 2i$

   $$= -4 + 2i - 2i$$
   $$= -4$$

In Example 1(b), notice that the sum of two complex numbers can be a real number.

Many of the properties of real numbers are valid for complex numbers as well. Here are some examples.

**REMARK** Rather than trying to memorize the multiplication rule at the right, you can simply remember how the Distributive Property is used to multiply two complex numbers. The procedure is similar to multiplying two polynomials and combining like terms.

**(a + bi)(c + di) = ac + adi + bci + bdi^2**

Distributive Property

$$= ac + (ad)i + (bc)i + (bd)i^2$$

Distributive Property

$$= ac + (ad)i + (bc)i + (bd)(-1)$$

Distributive Property

$$= ac - bd + (ad)i + (bc)i$$

Commutative Property

$$= (bc + ad)i + (ac - bd)$$

Associative Property

The procedure above is similar to multiplying two polynomials and combining like terms, as in the FOIL method.
Multiplying Complex Numbers

**EXAMPLE 2**

* a. \((3 + 2i)(3 - 2i) = 3(3 - 2i) + 2i(3 - 2i)\)  
  \(= 9 - 6i + 6i - 4i^2\)  
  \(= 9 - 6i + 6i - 4(-1)\)  
  \(= 9 + 4\)  
  \(= 13\)  
  Distributive Property  
  Simplify.  
  Write in standard form.

* b. \((3 + 2i)^2 = (3 + 2i)(3 + 2i)\)  
  \(= 3(3 + 2i) + 2i(3 + 2i)\)  
  \(= 9 + 6i + 6i + 4i^2\)  
  \(= 9 + 6i + 6i + 4(-1)\)  
  \(= 9 + 12i - 4\)  
  \(= 5 + 12i\)  
  Square of a binomial  
  Distributive Property  
  Simplify.  
  Write in standard form.

In Example 2(a), notice that the product of two complex numbers can be a real number. This occurs with pairs of complex numbers of the form \(a + bi\) and \(a - bi\), called complex conjugates.

\[(a + bi)(a - bi) = a^2 - abi + abi - b^2i^2\]
\[= a^2 - b^2(-1)\]
\[= a^2 + b^2\]

To write the quotient of \(a + bi\) and \(c + di\) in standard form, where \(c\) and \(d\) are not both zero, multiply the numerator and denominator by the complex conjugate of the denominator to obtain

\[\frac{a + bi}{c + di} = \frac{a + bi}{c + di} \cdot \frac{c - di}{c - di}\]
\[= \frac{(ac + bd) + (bc - ad)i}{c^2 + d^2}\]

Multiply numerator and denominator by complex conjugate of denominator.  
Write in standard form.

**EXAMPLE 3**

Writing Complex Numbers in Standard Form

\[\frac{2 + 3i}{4 - 2i} = \frac{2 + 3i}{4 - 2i} \cdot \frac{4 + 2i}{4 + 2i}\]
\[= \frac{8 + 4i + 12i + 6i^2}{16 - 4i^2}\]
\[= \frac{8 - 6 + 16i}{16 + 4}\]
\[= \frac{2 + 16i}{20}\]
\[= \frac{1}{10} + \frac{4}{5}i\]

Multiply numerator and denominator by complex conjugate of denominator.  
Expand.  
Simplify.  
Write in standard form.
Complex Solutions of Quadratic Equations

When using the Quadratic Formula to solve a quadratic equation, you often obtain a result such as \( \sqrt{-3} \), which you know is not a real number. By factoring out \( i \), you can write this number in standard form.

\[
\sqrt{-3} = \sqrt{3(-1)} = \sqrt{3} \sqrt{-1} = \sqrt{3}i
\]

The number \( \sqrt{3}i \) is called the principal square root of \( -3 \).

### Principal Square Root of a Negative Number

If \( a \) is a positive number, then the principal square root of the negative number \( -a \) is defined as

\[
\sqrt{-a} = \sqrt{a}i.
\]

### EXAMPLE 4 Writing Complex Numbers in Standard Form

**a.** \( \sqrt{-3} \sqrt{-12} = \sqrt{3} \sqrt{12}i \)

\[
= \sqrt{36}i^2
= 6(-1)
= -6
\]

**b.** \( \sqrt{-48} - \sqrt{-27} = \sqrt{48}i - \sqrt{27}i \)

\[
= 4\sqrt{3}i - 3\sqrt{3}i
= \sqrt{3}i
\]

**c.** \( (-1 + \sqrt{-3})^2 = (-1 + \sqrt{3})^2 \)

\[
= (-1)^2 - 2\sqrt{3}i + (\sqrt{3})^2(i^2)
= 1 - 2\sqrt{3}i + 3(-1)
= -2 - 2\sqrt{3}i
\]

### EXAMPLE 5 Complex Solutions of a Quadratic Equation

Solve \( 3x^2 - 2x + 5 = 0 \).

**Solution**

\[
x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(3)(5)}}{2(3)} \quad \text{Quadratic Formula}
\]

\[
= \frac{2 \pm \sqrt{-56}}{6}
= \frac{2 \pm 2\sqrt{14}i}{6}
= \frac{1}{3} \pm \frac{\sqrt{14}}{3}i \quad \text{Simplify.}
\]

**Write \( \sqrt{-56} \) in standard form.**

**Write in standard form.**
Appendix E  Complex Numbers

Trigonometric Form of a Complex Number

Just as real numbers can be represented by points on the real number line, you can represent a complex number as the point in a coordinate plane (the complex plane). The horizontal axis is called the real axis and the vertical axis is called the imaginary axis, as shown in Figure E.1.

The absolute value of a complex number $a + bi$ is defined as the distance between the origin and the point $(a, b)$.

The Absolute Value of a Complex Number

The absolute value of the complex number $z = a + bi$ is given by

$$ |a + bi| = \sqrt{a^2 + b^2}. $$

When the complex number $a + bi$ is a real number (that is, $b = 0$), this definition agrees with that given for the absolute value of a real number.

$$ |a + 0i| = \sqrt{a^2 + 0^2} = |a| $$

To work effectively with powers and roots of complex numbers, it is helpful to write complex numbers in trigonometric form. In Figure E.2, consider the nonzero complex number $z = a + bi$. By letting $\theta$ be the angle from the positive real axis (measured counterclockwise) to the line segment connecting the origin and the point $(a, b)$, you can write

$$ a = r \cos \theta \quad \text{and} \quad b = r \sin \theta $$

where $r = \sqrt{a^2 + b^2}$. Consequently, you have

$$ a + bi = (r \cos \theta) + (r \sin \theta)i $$

from which you can obtain the trigonometric form of a complex number.

Trigonometric Form of a Complex Number

The trigonometric form of the complex number $z = a + bi$ is given by

$$ z = r (\cos \theta + i \sin \theta) $$

where $a = r \cos \theta$, $b = r \sin \theta$, $r = \sqrt{a^2 + b^2}$, and $\tan \theta = b/a$. The number $r$ is the modulus of $z$, and $\theta$ is called an argument of $z$.

The trigonometric form of a complex number is also called the polar form. Because there are infinitely many choices for $\theta$, the trigonometric form of a complex number is not unique. Normally, $\theta$ is restricted to the interval $0 \leq \theta < 2\pi$, although on occasion it is convenient to use $\theta < 0$. 

Figure E.1

Figure E.2
EXAMPLE 6  Trigonometric Form of a Complex Number

Write the complex number \( z = -2 - 2\sqrt{3}i \) in trigonometric form.

Solution  The absolute value of \( z \) is

\[
    r = | -2 - 2\sqrt{3}i | = \sqrt{(-2)^2 + (-2\sqrt{3})^2} = \sqrt{16} = 4
\]

and the angle \( \theta \) is given by

\[
    \tan \theta = \frac{b}{a} = -\frac{2\sqrt{3}}{-2} = \sqrt{3}.
\]

Because \( \tan(\pi/3) = \sqrt{3} \) and because \( z = -2 - 2\sqrt{3}i \) lies in Quadrant III, choose \( \theta \) to be \( \theta = \pi + \pi/3 = 4\pi/3 \). So, the trigonometric form is

\[
    z = r(\cos \theta + i \sin \theta) = 4\left( \cos \frac{4\pi}{3} + i \sin \frac{4\pi}{3} \right).
\]

See Figure E.3.

The trigonometric form adapts nicely to multiplication and division of complex numbers. Consider the two complex numbers

\[
    z_1 = r_1(\cos \theta_1 + i \sin \theta_1) \quad \text{and} \quad z_2 = r_2(\cos \theta_2 + i \sin \theta_2).
\]

The product of \( z_1 \) and \( z_2 \) is

\[
    z_1z_2 = r_1r_2(\cos \theta_1 + i \sin \theta_1)(\cos \theta_2 + i \sin \theta_2) = r_1r_2[\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)].
\]

Using the sum and difference formulas for cosine and sine, you can rewrite this equation as

\[
    z_1z_2 = r_1r_2[\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)].
\]

This establishes the first part of the rule shown below. The second part is left for you to verify (see Exercise 109).

Product and Quotient of Two Complex Numbers

Let \( z_1 = r_1(\cos \theta_1 + i \sin \theta_1) \) and \( z_2 = r_2(\cos \theta_2 + i \sin \theta_2) \) be complex numbers.

\[
    z_1z_2 = r_1r_2[\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)] \quad \text{Product}
\]

\[
    \frac{z_1}{z_2} = \frac{r_1}{r_2} \left[ \cos(\theta_1 - \theta_2) + i \sin(\theta_1 - \theta_2) \right], \quad z_2 \neq 0 \quad \text{Quotient}
\]
Note that this rule says that to multiply two complex numbers you multiply moduli and add arguments, whereas to divide two complex numbers you divide moduli and subtract arguments.

**EXAMPLE 7  Multiplying Complex Numbers**

Find the product $z_1z_2$ of the complex numbers.

$$z_1 = 2\left(\cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3}\right), \quad z_2 = 8\left(\cos \frac{11\pi}{6} + i \sin \frac{11\pi}{6}\right)$$

**Solution**

$$z_1z_2 = 2\left(\cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3}\right) \cdot 8\left(\cos \frac{11\pi}{6} + i \sin \frac{11\pi}{6}\right)$$

$$= 16\left[\cos \left(\frac{2\pi}{3} + \frac{11\pi}{6}\right) + i \sin \left(\frac{2\pi}{3} + \frac{11\pi}{6}\right)\right]$$

$$= 16\left[\cos \frac{5\pi}{2} + i \sin \frac{5\pi}{2}\right]$$

$$= 16\left[\cos \frac{\pi}{2} + i \sin \frac{\pi}{2}\right]$$

$$= 16[0 + i(1)]$$

$$= 16i$$

Check this result by first converting to the standard forms $z_1 = -1 + \sqrt{3}i$ and $z_2 = 4\sqrt{3} - 4i$ and then multiplying algebraically.

**EXAMPLE 8  Dividing Complex Numbers**

Find the quotient $z_1/z_2$ of the complex numbers.

$$z_1 = 24(\cos 300^\circ + i \sin 300^\circ), \quad z_2 = 8(\cos 75^\circ + i \sin 75^\circ)$$

**Solution**

$$\frac{z_1}{z_2} = \frac{24(\cos 300^\circ + i \sin 300^\circ)}{8(\cos 75^\circ + i \sin 75^\circ)}$$

$$= \frac{24}{8}\left[\cos(300^\circ - 75^\circ) + i \sin(300^\circ - 75^\circ)\right]$$

$$= 3[\cos 225^\circ + i \sin 225^\circ]$$

$$= 3\left(\frac{-\sqrt{2}}{2} + i \left(\frac{-\sqrt{2}}{2}\right)\right)$$

$$= -\frac{3\sqrt{2}}{2} - \frac{3\sqrt{2}}{2}i$$
Powers and Roots of Complex Numbers

To raise a complex number to a power, consider repeated use of the multiplication rule.

\[ z = r(\cos \theta + i \sin \theta) \]
\[ z^2 = r^2(\cos 2\theta + i \sin 2\theta) \]
\[ z^3 = r^3(\cos 3\theta + i \sin 3\theta) \]

This pattern leads to the next theorem, which is named after the French mathematician Abraham DeMoivre (1667–1754).

**Theorem E.1 DeMoivre’s Theorem**

If \( z = r(\cos \theta + i \sin \theta) \) is a complex number and \( n \) is a positive integer, then

\[ z^n = [r(\cos \theta + i \sin \theta)]^n = r^n(\cos n\theta + i \sin n\theta). \]

**Example 9 Finding Powers of a Complex Number**

Use DeMoivre’s Theorem to find \((-1 + \sqrt{3}i)^{12}\).

**Solution**  First convert the complex number to trigonometric form.

\[-1 + \sqrt{3}i = 2 \left( \cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3} \right)\]

Then, by DeMoivre’s Theorem, you have

\[ (-1 + \sqrt{3}i)^{12} = \left[ 2 \left( \cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3} \right) \right]^{12} \]
\[ = 2^{12} \left[ \cos \left( 12 \cdot \frac{2\pi}{3} \right) + i \sin \left( 12 \cdot \frac{2\pi}{3} \right) \right] \]
\[ = 4096(\cos 8\pi + i \sin 8\pi) \]
\[ = 4096. \]

Recall that a consequence of the Fundamental Theorem of Algebra is that a polynomial equation of degree \( n \) has \( n \) solutions in the complex number system. Each solution is an \( n \)th root of the equation. The \( n \)th root of a complex number is defined below.

**Definition of \( n \)th Root of a Complex Number**

The complex number \( u = a + bi \) is an \( n \)th root of the complex number \( z \) when

\[ z = u^n = (a + bi)^n. \]
To find a formula for an $n$th root of a complex number, let $u$ be an $n$th root of $z$, where

$$u = s (\cos \beta + i \sin \beta) \quad \text{and} \quad z = r (\cos \theta + i \sin \theta).$$

By DeMoivre’s Theorem and the fact that $u^n = z$, you have

$$s^n (\cos n\beta + i \sin n\beta) = r (\cos \theta + i \sin \theta).$$

Taking the absolute value of each side of this equation, it follows that $s^n = r$. Substituting $s^n$ for $r$ in the previous equation and dividing by $s^n$, you get

$$\cos n\beta + i \sin n\beta = \cos \theta + i \sin \theta.$$

So, it follows that

$$\cos n\beta = \cos \theta \quad \text{and} \quad \sin n\beta = \sin \theta.$$

Because both sine and cosine have a period of $2\pi$, these last two equations have solutions if and only if the angles differ by a multiple of $2\pi$. Consequently, there must exist an integer $k$ such that

$$n\beta = \theta + 2\pi k$$
$$\beta = \frac{\theta + 2\pi k}{n}.$$  

By substituting this value for $\beta$ into the trigonometric form of $u$, you get the result stated in the next theorem.

**Theorem E.2  nth Roots of a Complex Number**

For a positive integer $n$, the complex number $z = r(\cos \theta + i \sin \theta)$ has exactly $n$ distinct $n$th roots given by

$$\sqrt[n]{r} \left( \cos \left( \frac{\theta + 2\pi k}{n} \right) + i \sin \left( \frac{\theta + 2\pi k}{n} \right) \right)$$

where $k = 0, 1, 2, \ldots, n - 1$.

For $k > n - 1$, the roots begin to repeat. For instance, when $k = n$, the angle

$$\frac{\theta + 2\pi n}{n} = \frac{\theta + 2\pi}{n}$$

is coterminal with $\theta/n$, which is also obtained when $k = 0$.

The formula for the $n$th roots of a complex number $z$ has a nice geometric interpretation, as shown in Figure E.4. Note that because the $n$th roots of $z$ all have the same magnitude $\sqrt[n]{r}$, they all lie on a circle of radius $\sqrt[n]{r}$ with center at the origin. Furthermore, because successive $n$th roots have arguments that differ by $2\pi/n$, the $n$ roots are equally spaced along the circle.

**Figure E.4**
E10 Appendix E Complex Numbers

EXAMPLE 10 Finding the nth Roots of a Complex Number

Find the three cube roots of \( z = -2 + 2i \).

**Solution** Because \( z \) lies in Quadrant II, the trigonometric form for \( z \) is

\[
z = -2 + 2i = \sqrt{8} \left( \cos 135^\circ + i \sin 135^\circ \right).
\]

By the formula for \( n \)th roots, the cube roots have the form

\[
\sqrt[3]{8} \left( \cos \frac{135^\circ + 360^\circ k}{3} + i \sin \frac{135^\circ + 360^\circ k}{3} \right).
\]

Finally, for \( k = 0, 1, \) and \( 2 \), you obtain the roots

\[
\sqrt[3]{2} \left( \cos 45^\circ + i \sin 45^\circ \right) = 1 + i
\]

\[
\sqrt[3]{2} \left( \cos 165^\circ + i \sin 165^\circ \right) = -1.3660 + 0.3660i
\]

\[
\sqrt[3]{2} \left( \cos 285^\circ + i \sin 285^\circ \right) = 0.3660 - 1.3660i.
\]

E Exercises

Performing Operations In Exercises 1–24, perform the operation and write the result in standard form.

1. \((5 + i) + (6 - 2i)\)
2. \((13 - 2i) + (-5 + 6i)\)
3. \((8 - i) - (4 - i)\)
4. \((3 + 2i) - (6 + 13i)\)
5. \((-2 + \sqrt{-8}) + (5 - \sqrt{-56})\)
6. \((8 + \sqrt{-18}) - (4 + 3\sqrt{2})\)
7. \(13i - (14 - 7i)\)
8. \(22 + (-5 + 8i) + 10i\)
9. \(-\left(\frac{1}{2} + \frac{i}{3}\right) + \left(\frac{2}{3} + \frac{4}{5}i\right)\)
10. \((1.6 + 3.2i) + (-5.8 + 4.3i)\)
11. \(\sqrt{-6} \cdot \sqrt{-3}\)
12. \(\sqrt{-2} \cdot \sqrt{-10}\)
13. \((\sqrt{-10})^2\)
14. \((\sqrt{-75})^2\)
15. \((1 + i)(3 - 2i)\)
16. \((6 - 2i)(2 - 3i)\)
17. \(6i(5 - 2i)\)
18. \(-8i(9 + 4i)\)
19. \(\left(\sqrt{14} + \sqrt{10i}\right)\left(\sqrt{14} - \sqrt{10i}\right)\)
20. \(\left(3 + \sqrt{-5}\right)(7 - \sqrt{-10})\)
21. \((4 + 5i)^2\)
22. \((2 - 3i)^2\)
23. \((2 + 3i)^2 + (2 - 3i)^2\)
24. \((1 - 2i)^2 - (1 + 2i)^2\)

Writing a Complex Conjugate In Exercises 25–32, write the complex conjugate of the complex number. Then multiply the number by its complex conjugate.

25. \(5 + 3i\)
26. \(9 - 12i\)
27. \(-2 - \sqrt{2}i\)
28. \(-4 + \sqrt{2}i\)
29. \(20i\)
30. \(\sqrt{-15}\)
31. \(\sqrt{8}\)
32. \(1 + \sqrt{8}\)

Writing in Standard Form In Exercises 33–42, write the quotient in standard form.

33. \(\frac{6}{i}\)
34. \(\frac{10}{2i}\)
35. \(\frac{4}{4 - 5i}\)
36. \(\frac{3}{1 - i}\)
37. \(\frac{2 + i}{2 - i}\)
38. \(\frac{8 - 7i}{1 - 2i}\)
39. \(\frac{6 - 7i}{i}\)
40. \(\frac{8 + 20i}{2i}\)
41. \(\frac{1}{(4 - 3i)^2}\)
42. \(\frac{(2 - 3i)(5i)}{2 + 3i}\)

Performing Operations In Exercises 43–46, perform the operation and write the result in standard form.

43. \(\frac{2}{1 + i} - \frac{3}{1 - i}\)
44. \(\frac{2i}{2 + i} + \frac{5}{2 - i}\)
45. \(\frac{i}{3 - 2i} + \frac{2i}{3 + 8i}\)
46. \(\frac{1 + i}{i} - \frac{3}{4 - i}\)
Using the Quadratic Formula  In Exercises 47–54, use the Quadratic Formula to solve the quadratic equation.

47. \( x^2 - 2x + 2 = 0 \)
48. \( x^2 + 6x + 10 = 0 \)
49. \( 4x^2 + 16x + 17 = 0 \)
50. \( 9x^2 - 6x + 37 = 0 \)
51. \( 4x^2 + 16x + 15 = 0 \)
52. \( 9x^2 - 6x - 35 = 0 \)
53. \( 16x^2 - 4x + 3 = 0 \)
54. \( 5x^2 + 6x + 3 = 0 \)

Writing in Standard Form  In Exercises 55–62, simplify the complex number and write it in standard form.

55. \( -6i + i^2 \)
56. \( 4i^2 - 2i^3 \)
57. \( -5i^5 \)
58. \( (-i)^3 \)
59. \( (\sqrt{-73})^i \)
60. \( (\sqrt{-2})^6 \)
61. \( \frac{1}{17} \)
62. \( \frac{1}{(2i)^7} \)

Absolute Value of a Complex Number  In Exercises 63–68, plot the complex number and find its absolute value.

63. \( -5i \)
64. \( -5 \)
65. \( -4 + 4i \)
66. \( 5 - 12i \)
67. \( 6 - 7i \)
68. \( -8 + 3i \)

Writing in Trigonometric Form  In Exercises 69–76, represent the complex number graphically, and find the trigonometric form of the number.

69. \( 3 - 3i \)
70. \( 2 + 2i \)
71. \( \sqrt{3} + i \)
72. \( -1 + \sqrt{3}i \)
73. \( -2(1 + \sqrt{3}i) \)
74. \( \frac{2}{\sqrt{3} - i} \)
75. \( 6i \)
76. \( 6 \sqrt{2} \)

Writing in Standard Form  In Exercises 77–82, represent the complex number graphically, and find the standard form of the number.

77. \( 2(\cos 150^\circ + i \sin 150^\circ) \)
78. \( 5(\cos 135^\circ + i \sin 135^\circ) \)
79. \( 2(\cos 300^\circ + i \sin 300^\circ) \)
80. \( \frac{2}{3}(\cos 315^\circ + i \sin 315^\circ) \)
81. \( 3.75 \left( \cos \frac{2\pi}{4} + i \sin \frac{2\pi}{4} \right) \)
82. \( 8 \left( \cos \frac{\pi}{12} + i \sin \frac{\pi}{12} \right) \)

Performing Operations  In Exercises 83–86, perform the operation and leave the result in trigonometric form.

83. \( \left[ 3 \left( \cos \frac{\pi}{3} + i \sin \frac{\pi}{3} \right) \right] \left[ 4 \left( \cos \frac{\pi}{6} + i \sin \frac{\pi}{6} \right) \right] \)
84. \( \frac{3}{4} \left( \cos \frac{\pi}{2} + i \sin \frac{\pi}{2} \right) \left[ 4 \left( \cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \right) \right] \)
85. \( \frac{1}{2} \left( \cos 140^\circ + i \sin 140^\circ \right) \left( \cos 60^\circ + i \sin 60^\circ \right) \)
86. \( \frac{\cos (5\pi/3) + i \sin (5\pi/3)}{\cos \pi + i \sin \pi} \)

Using DeMoivre’s Theorem  In Exercises 87–94, use DeMoivre’s Theorem to find the indicated power of the complex number. Write the result in standard form.

87. \( (1 + i)^5 \)
88. \( (2 + 2i)^6 \)
89. \( (-1 + i)^{10} \)
90. \( (1 - i)^{12} \)
91. \( 2(\sqrt{3} + i)^7 \)
92. \( 4(1 - \sqrt{3}i)^9 \)
93. \( \left( \frac{\cos \frac{5\pi}{4} + i \sin \frac{5\pi}{4}}{4} \right)^{10} \)
94. \( \left( \frac{\cos \frac{\pi}{2} + i \sin \frac{\pi}{2}}{2} \right)^{10} \)

Finding nth Roots  In Exercises 95–100, (a) use Theorem E.2 to find the indicated roots of the complex number, (b) represent each of the roots graphically, and (c) write each of the roots in standard form.

95. Square roots of \( 5(\cos 120^\circ + i \sin 120^\circ) \)
96. Square roots of \( 16(\cos 60^\circ + i \sin 60^\circ) \)
97. Fourth roots of \( 16(\cos \frac{4\pi}{3} + i \sin \frac{4\pi}{3}) \)
98. Fifth roots of \( 32(\cos \frac{5\pi}{6} + i \sin \frac{5\pi}{6}) \)
99. Cube roots of \( -\frac{125}{2}(1 + \sin \frac{\pi}{6}) \)
100. Cube roots of \( -4\sqrt{2}(1 - i) \)

Solving an Equation  In Exercises 101–108, use Theorem E.2 to find all the solutions of the equation and represent the solutions graphically.

101. \( x^4 - i = 0 \)
102. \( x^4 + 1 = 0 \)
103. \( x^3 + 243 = 0 \)
104. \( x^4 - 81 = 0 \)
105. \( x^3 + 64i = 0 \)
106. \( x^4 - 64i = 0 \)
107. \( x^4 - (1 - i) = 0 \)
108. \( x^4 + (1 + i) = 0 \)

109. Proof  Given two complex numbers

\[ z_1 = r_1(\cos \theta_1 + i \sin \theta_1) \quad \text{and} \quad z_2 = r_2(\cos \theta_2 + i \sin \theta_2) \]

show that

\[ \frac{z_1}{z_2} = \frac{r_1}{r_2}(\cos(\theta_1 - \theta_2) + i \sin(\theta_1 - \theta_2)) \]

\[ z_2 \neq 0. \]